6-4 Polynomial Functions

Main Ideas

- Evaluate polynomial functions.
- Identify general shapes of graphs of polynomial functions.

New Vocabulary

polynomial in one variable
leading coefficient
polynomial function
end behavior

GET READY for the Lesson

A cross section of a honeycomb has a pattern with one hexagon surrounded by six more hexagons. Surrounding these is a third ring of 12 hexagons, and so on. The total number of hexagons in a honeycomb can be modeled by the function $f(r) = 3r^2 - 3r + 1$, where r is the number of rings and $f(r)$ is the number of hexagons.

Polynomial Functions The expression $3r^2 - 3r + 1$ is a **polynomial in one variable** since it only contains one variable, r.

KEY CONCEPT — Polynomial in One Variable

Words A polynomial of degree n in one variable x is an expression of the form $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$, where the coefficients $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ represent real numbers, a_n is not zero, and n represents a nonnegative integer.

Example $3x^5 + 2x^4 - 5x^3 + x^2 + 1$
$n = 5, a_5 = 3, a_4 = 2, a_3 = -5, a_2 = 1, a_1 = 0$, and $a_0 = 1$

The degree of a polynomial in one variable is the greatest exponent of its variable. The **leading coefficient** is the coefficient of the term with the highest degree.

Polynomial	Expression	Degree	Leading Coefficient
Constant	9	0	9
Linear	$x - 2$	1	1
Quadratic	$3x^2 + 4x - 5$	2	3
Cubic	$4x^3 - 6$	3	4
General	$a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$	n	a_n

EXAMPLE Find Degrees and Leading Coefficients

1 **State the degree and leading coefficient of each polynomial in one variable. If it is not a polynomial in one variable, explain why.**

a. $7x^4 + 5x^2 + x - 9$

This is a polynomial in one variable.

The degree is 4, and the leading coefficient is 7.

(continued on the next page)

b. $8x^2 + 3xy - 2y^2$

This is not a polynomial in one variable. It contains two variables, x and y.

CHECK Your Progress

1A. $7x^6 - 4x^3 + \frac{1}{x}$ **1B.** $\frac{1}{2}x^2 + 2x^3 - x^5$

A polynomial equation used to represent a function is called a **polynomial function**. For example, the equation $f(x) = 4x^2 - 5x + 2$ is a quadratic polynomial function, and the equation $p(x) = 2x^3 + 4x^2 - 5x + 7$ is a cubic polynomial function. Other polynomial functions can be defined by the following general rule.

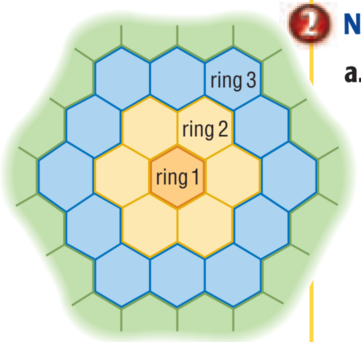
KEY CONCEPT ***Definition of a Polynomial Function***

Words A polynomial function of degree n is a continuous function that can be described by an equation of the form $P(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$, where the coefficients $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ represent real numbers, a_n is not zero, and n represents a nonnegative integer.

Example $f(x) = 4x^2 - 3x + 2$
$n = 2, a_2 = 4, a_1 = -3, a_0 = 2$

If you know an element in the domain of any polynomial function, you can find the corresponding value in the range. Recall that $f(3)$ can be found by evaluating the function for $x = 3$.

Real-World EXAMPLE



Rings of a Honeycomb

2 **NATURE** **Refer to the application at the beginning of the lesson.**

a. Show that the polynomial function $f(r) = 3r^2 - 3r + 1$ gives the total number of hexagons when $r = 1$, 2, and 3.

Find the values of $f(1)$, $f(2)$, and $f(3)$.

$f(r) = 3r^2 - 3r + 1$	$f(r) = 3r^2 - 3r + 1$	$f(r) = 3r^2 - 3r + 1$
$f(1) = 3(1)^2 - 3(1) + 1$	$f(2) = 3(2)^2 - 3(2) + 1$	$f(3) = 3(3)^2 - 3(3) + 1$
$= 3 - 3 + 1$ or 1	$= 12 - 6 + 1$ or 7	$= 27 - 9 + 1$ or 19

You know the numbers of hexagons in the first three rings are 1, 6, and 12. So, the total number of hexagons with one ring is 1, two rings is 6 + 1 or 7, and three rings is 12 + 6 + 1 or 19. These match the functional values for $r = 1$, 2, and 3, respectively. That is 1, 7, and 19 are the range values corresponding to the domain values of 1, 2, and 3.

b. Find the total number of hexagons in a honeycomb with 12 rings.

$f(r) = 3r^2 - 3r + 1$ Original function

$f(12) = 3(12)^2 - 3(12) + 1$ Replace r with 12.

$= 432 - 36 + 1$ or 397 Simplify.

CHECK Your Progress

2A. Show that $f(r)$ gives the total number of hexagons when $r = 4$.

2B. Find the total number of hexagons in a honeycomb with 20 rings.

You can also evaluate functions for variables and algebraic expressions.

EXAMPLE 3 Function Values of Variables

Find $q(a + 1) - 2q(a)$ if $q(x) = x^2 + 3x + 4$.

To evaluate $q(a + 1)$, replace x in $q(x)$ with $a + 1$.

$q(x) = x^2 + 3x + 4$	Original function
$q(a + 1) = (a + 1)^2 + 3(a + 1) + 4$	Replace x with $a + 1$.
$= a^2 + 2a + 1 + 3a + 3 + 4$	Simplify $(a + 1)^2$ and $3(a + 1)$.
$= a^2 + 5a + 8$	Simplify.

To evaluate $2q(a)$, replace x with a in $q(x)$, then multiply the expression by 2.

$q(x) = x^2 + 3x + 4$	Original function
$2q(a) = 2(a^2 + 3a + 4)$	Replace x with a.
$= 2a^2 + 6a + 8$	Distributive Property

Now evaluate $q(a + 1) - 2q(a)$.

$q(a + 1) - 2q(a) = a^2 + 5a + 8 - (2a^2 + 6a + 8)$	Replace $q(a + 1)$ and $2q(a)$.
$= a^2 + 5a + 8 - 2a^2 - 6a - 8$	
$= -a^2 - a$	Simplify.

Study Tip

Function Values

When finding function values of expressions, be sure to take note of where the coefficients occur. In Example 3, $2q(a)$ is 2 times the function value of a, not $q(2a)$, the function value of $2a$.

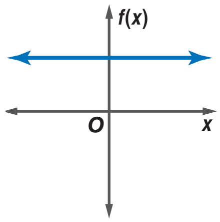
CHECK Your Progress

3A. Find $f(b^2)$ if $f(x) = 2x^2 + 3x - 1$.

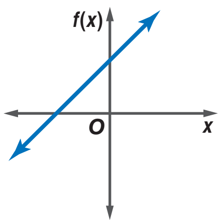
3B. Find $2g(c + 2) + 3g(2c)$ if $g(x) = x^2 - 4$.

Graphs of Polynomial Functions The general shapes of the graphs of several polynomial functions are shown below. These graphs show the *maximum* number of times the graph of each type of polynomial may intersect the x-axis. Recall that the x-coordinate of the point at which the graph intersects the x-axis is called a *zero* of a function. How does the degree compare to the maximum number of real zeros?

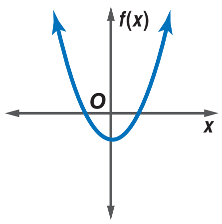
Constant function
Degree 0



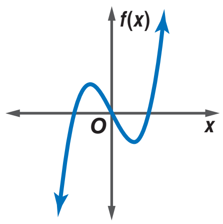
Linear function
Degree 1



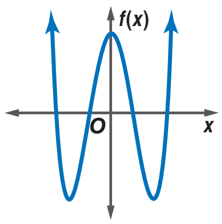
Quadratic function
Degree 2



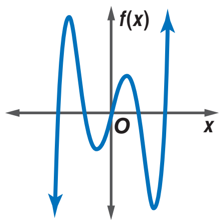
Cubic function
Degree 3



Quartic function
Degree 4



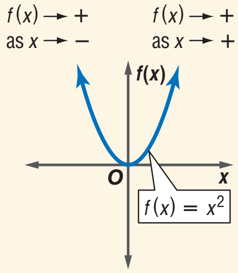
Quintic function
Degree 5



The **end behavior** is the behavior of the graph as x approaches positive infinity ($+\infty$) or negative infinity ($-\infty$). This is represented as $x \rightarrow +\infty$ and $x \rightarrow -\infty$, respectively. *$x \rightarrow +\infty$ is read x approaches positive infinity.* Notice the shapes of the graphs for even-degree polynomial functions and odd-degree polynomial functions. The degree and leading coefficient of a polynomial function determine the graph's end behavior.

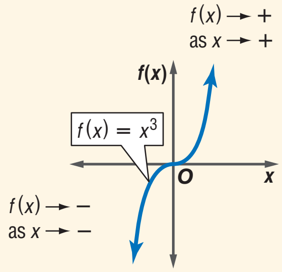
CONCEPT SUMMARY — *End Behavior of a Polynomial Function*

Degree: even
Leading Coefficient: positive
End Behavior:



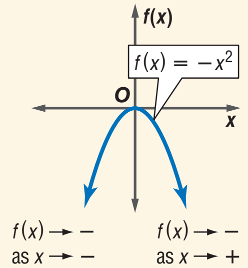
Domain: all reals
Range: all reals $\geq$ minimum

Degree: odd
Leading Coefficient: positive
End Behavior:



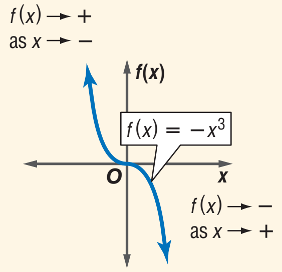
Domain: all reals
Range: all reals

Degree: even
Leading Coefficient: negative
End Behavior:



Domain: all reals
Range: all reals $\leq$ maximum

Degree: odd
Leading Coefficient: negative
End Behavior:



Domain: all reals
Range: all reals

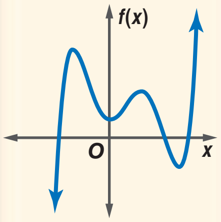
For any polynomial function, the domain is all real numbers. For any polynomial function of odd degree, the range is all real numbers. For polynomial functions of even degree, the range is all real numbers greater than or equal to some number or all real numbers less than or equal to some number; it is never all real numbers.

The graph of an even-degree function may or may not intersect the x-axis. If it intersects the x-axis in two places, the function has two real zeros. If it does not intersect the x-axis, the roots of the related equation are imaginary and cannot be determined from the graph. If the graph is tangent to the x-axis, as shown above, there are two zeros that are the same number. The graph of an odd-degree function always crosses the x-axis at least once, and thus the function always has at least one real zero.

Study Tip

Number of Zeros

The number of zeros of an odd-degree function may be less than the maximum by a multiple of 2. For example, the graph of a quintic function may only cross the x-axis 1, 3, or 5 times.

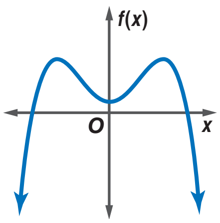


The same is true for an even-degree function. One exception is when the graph of $f(x)$ touches the x-axis.

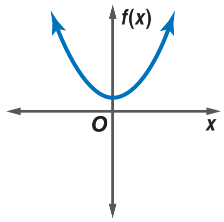
EXAMPLE 4 Graphs of Polynomial Functions

For each graph,
- **describe the end behavior,**
- **determine whether it represents an odd-degree or an even-degree polynomial function, and**
- **state the number of real zeros.**

a.



b.

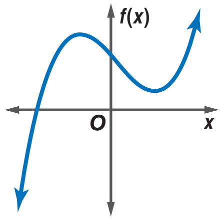


a. • $f(x) \to -\infty$ as $x \to +\infty$. $f(x) \to -\infty$ as $x \to -\infty$.
- It is an even-degree polynomial function.
- The graph intersects the x-axis at two points, so the function has two real zeros.

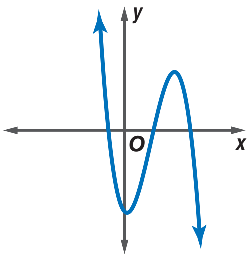
b. • $f(x) \to +\infty$ as $x \to +\infty$. $f(x) \to +\infty$ as $x \to -\infty$.
- It is an even-degree polynomial function.
- This graph does not intersect the x-axis, so the function has no real zeros.

CHECK Your Progress

4A.



4B.



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CHECK Your Understanding

Example 1 (pp. 331–332)

State the degree and leading coefficient of each polynomial in one variable. If it is not a polynomial in one variable, explain why.

1. $5x^6 - 8x^2$

2. $2b + 4b^3 - 3b^5 - 7$

Example 2 (p. 332)

Find $p(3)$ and $p(-1)$ for each function.

3. $p(x) = -x^3 + x^2 - x$

4. $p(x) = x^4 - 3x^3 + 2x^2 - 5x + 1$

5. BIOLOGY The intensity of light emitted by a firefly can be determined by $L(t) = 10 + 0.3t + 0.4t^2 - 0.01t^3$, where t is temperature in degrees Celsius and $L(t)$ is light intensity in lumens. If the temperature is 30°C, find the light intensity.

Example 3 (p. 333)

If $p(x) = 2x^3 + 6x - 12$ and $q(x) = 5x^2 + 4$, find each value.

6. $p(a^3)$

7. $5[q(2a)]$

8. $3p(a) - q(a + 1)$

Example 4 (pp. 334–335)

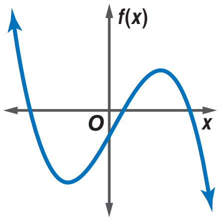
For each graph,

a. **describe the end behavior,**

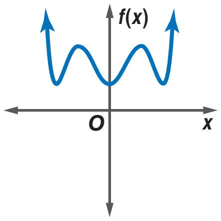
b. **determine whether it represents an odd-degree or an even-degree polynomial function, and**

c. **state the number of real zeros.**

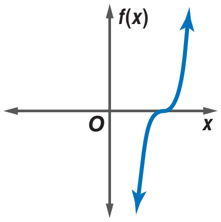
9.



10.



11.



Exercises

HOMEWORK HELP	
For Exercises	See Examples
12–17	1
18–21, 34, 35	2
22–27	3
28–33	4

State the degree and leading coefficient of each polynomial in one variable. If it is not a polynomial in one variable, explain why.

12. $7 - x$

13. $(a + 1)(a^2 - 4)$

14. $a^2 + 2ab + b^2$

15. $c^2 + c - \frac{1}{c}$

16. $6x^4 + 3x^2 + 4x - 8$

17. $7 + 3x^2 - 5x^3 + 6x^2 - 2x$

Find $p(4)$ and $p(-2)$ for each function.

18. $p(x) = 2 - x$

19. $p(x) = x^2 - 3x + 8$

20. $p(x) = 2x^3 - x^2 + 5x - 7$

21. $p(x) = x^5 - x^2$

If $p(x) = 3x^2 - 2x + 5$ and $r(x) = x^3 + x + 1$, find each value.

22. $r(3a)$

23. $4p(a)$

24. $p(a^2)$

25. $p(2a^3)$

26. $r(x + 1)$

27. $p(x^2 + 3)$

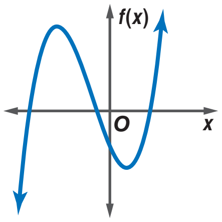
For each graph,

a. describe the end behavior,

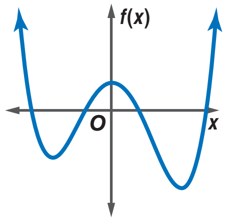
b. determine whether it represents an odd-degree or an even-degree polynomial function, and

c. state the number of real zeros.

28.



29.



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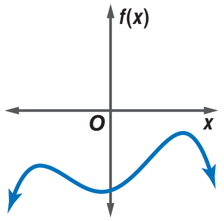
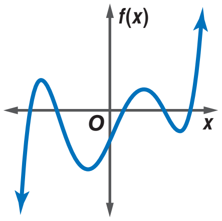


31.



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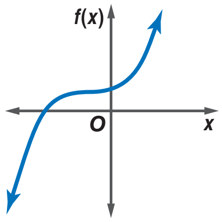
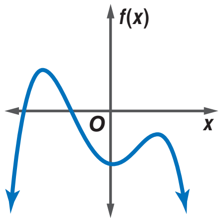


33.



34. ENERGY The power generated by a windmill is a function of the speed of the wind. The approximate power is given by the function $P(s) = \frac{s^3}{1000}$, where s represents the speed of the wind in kilometers per hour. Find the units of power $P(s)$ generated by a windmill when the wind speed is 18 kilometers per hour.

35. PHYSICS For a moving object with mass m in kilograms, the kinetic energy KE in joules is given by the function $KE(v) = \frac{1}{2}mv^2$, where v represents the speed of the object in meters per second. Find the kinetic energy of an all-terrain vehicle with a mass of 171 kilograms moving at a speed of 11 meters/second.

Find $p(4)$ and $p(-2)$ for each function.

36. $p(x) = x^4 - 7x^3 + 8x - 6$

37. $p(x) = 7x^2 - 9x + 10$

38. $p(x) = \frac{1}{2}x^4 - 2x^2 + 4$

39. $p(x) = \frac{1}{8}x^3 - \frac{1}{4}x^2 - \frac{1}{2}x + 5$

If $p(x) = 3x^2 - 2x + 5$ and $r(x) = x^3 + x + 1$, find each value.

40. $2[p(x + 4)]$

41. $r(x + 1) - r(x^2)$

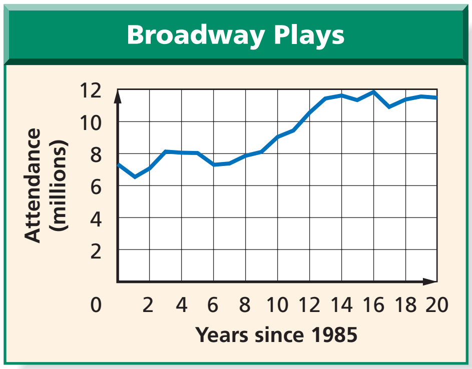
42. $3[p(x^2 - 1)] + 4p(x)$

Real-World Link

The Phantom of the Opera is the longest-running Broadway show in history.

Source: playbill.com

THEATER **For Exercises 43–45, use the graph that models the attendance at Broadway plays (in millions) from 1985–2005.**



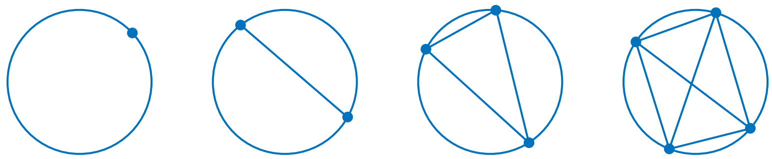
43. Is the graph an odd-degree or even-degree function?

44. Discuss the end behavior.

45. Do you think attendance at Broadway plays will increase or decrease after 2005? Explain your reasoning.

PATTERNS **For Exercises 46–48, use the diagrams below that show the maximum number of regions formed by connecting points on a circle.**

1 point, 1 region | 2 points, 2 regions | 3 points, 4 regions | 4 points, 8 regions



46. The number of regions formed by connecting n points of a circle can be described by the function $f(n) = \frac{1}{24}(n^4 - 6n^3 + 23n^2 - 18n + 24)$. What is the degree of this polynomial function?

47. Find the number of regions formed by connecting 5 points of a circle. Draw a diagram to verify your solution.

48. How many points would you have to connect to form 99 regions?

EXTRA PRACTICE
See pages 903, 931.

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H.O.T. Problems

49. **REASONING** Explain why a constant polynomial such as $f(x) = 4$ has degree 0 and a linear polynomial such as $f(x) = x + 5$ has degree 1.

50. **OPEN ENDED** Sketch the graph of an odd-degree polynomial function with a negative leading coefficient and three real roots.

51. **REASONING** Determine whether the following statement is *always*, *sometimes* or *never* true. Explain.

A polynomial function that has four real roots is a fourth-degree polynomial.

CHALLENGE **For Exercises 52–55, use the following information.**
The graph of the polynomial function $f(x) = ax(x - 4)(x + 1)$ goes through the point at (5, 15).

52. Find the value of a.

53. For what value(s) of x will $f(x) = 0$?

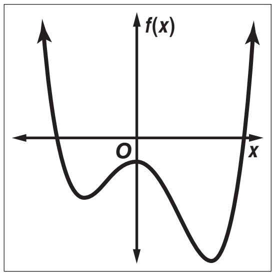
54. Simplify and rewrite the function as a cubic function.

55. Sketch the graph of the function.

56. ***Writing in Math*** Use the information on page 331 to explain where polynomial functions are found in nature. Include an explanation of how you could use the equation to find the number of hexagons in the tenth ring and any other examples of patterns found in nature that might be modeled by a polynomial equation.

STANDARDIZED TEST PRACTICE

57. ACT/SAT The figure at the right shows the graph of a polynomial function $f(x)$. Which of the following could be the degree of $f(x)$?



A 2

B 3

C 4

D 5

58. REVIEW Which polynomial represents $(4x^2 + 5x - 3)(2x - 7)$?

F $8x^3 - 18x^2 - 41x - 21$

G $8x^3 + 18x^2 + 29x - 21$

H $8x^3 - 18x^2 - 41x + 21$

J $8x^3 + 18x^2 - 29x + 21$

Spiral Review

Simplify. (Lesson 6-3)

59. $(t^3 - 3t + 2) \div (t + 2)$

60. $(y^2 + 4y + 3)(y + 1)^{-1}$

61. $\dfrac{x^3 - 3x^2 + 2x - 6}{x - 3}$

62. $\dfrac{3x^4 + x^3 - 8x^2 + 10x - 3}{3x - 2}$

63. BUSINESS Ms. Schifflet is writing a computer program to find the salaries of her employees after their annual raise. The percent of increase is represented by p. Marty's salary is \$23,450 now. Write a polynomial to represent Marty's salary in one year and another to represent Marty's salary after three years. Assume that the rate of increase will be the same for each of the three years. (Lesson 6-2)

Solve each equation by completing the square. (Lesson 5-5)

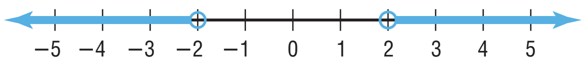
64. $x^2 - 8x - 2 = 0$

65. $x^2 + \frac{1}{3}x - \frac{35}{36} = 0$

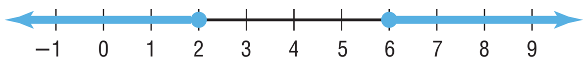
Write an absolute value inequality for each graph. (Lesson 1-6)

66.

67.



68.



69.

Name the property illustrated by each statement. (Lesson 1-3)

70. If $3x = 4y$ and $4y = 15z$, then $3x = 15z$.

71. $5y(4a - 6b) = 20ay - 30by$

72. $2 + (3 + x) = (2 + 3) + x$

GET READY for the Next Lesson

PREREQUISITE SKILL **Graph each equation by making a table of values.** (Lesson 5-1)

73. $y = x^2 + 4$

74. $y = -x^2 + 6x - 5$

75. $y = \frac{1}{2}x^2 + 2x - 6$